Symmetrical Buckling of Right-Angled Isosceles Triangular Plates

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Summary: The problem considered is that of the buckling of a right-angled isosceles triangular plate subjected to shear along the two perpendicular edges together with uniform compression in all directions. Four combinations of boundary conditions are considered, and the buckle is assumed to be symmetrical about the bisector of the right angle. In each case it is shown that, if the plate is subjected to shear only, the critical stress is changed considerably upon reversal of the shear. The larger value occurs when the shear is tending to decrease the right angle. Because of this fact, the interaction curve, giving critical combinations of compression and shear, is unsymmetrical and the critical compressive stress can be appreciably increased by the application of a suitable amount of shear.

1. Introduction

In a recent paper[1] the author considered the problem of the buckling of a clamped oblique plate (*i.e.* in the shape of a parallelogram) under pure shear parallel and perpendicular to two edges of the plate. It was found that the critical stress when the shear is tending to increase the obliquity of the plate is less than that corresponding to the reverse direction. It was also shown that for a 45° oblique plate the ratio between the two critical shear stresses lies between 1 and 2·69 if the edges are clamped, and between 1 and 8 if the edges are simply-supported, the actual value depending on the side ratio of the parallelogram.

A previous paper[2] derived numerical results for the critical stresses of clamped oblique plates under compression parallel to two of the sides, and the question now arises as to the shape of the interaction curve, giving critical combinations of compression and shear.

In a rectangular plate a reversal of the shear has no effect on its critical value, and the interaction curve is symmetrical about the axis of compressive stress. The curve is approximately parabolic, and the application of any shear stress reduces the critical compressive stress. For an oblique plate, however, the interaction curve is no longer symmetrical and it is therefore possible that the application of a certain amount of shear stress in a direction tending to decrease the obliquity of the plate may actually increase the critical compressive stress.

The computation involved in the oblique plate problem is very tedious and in order to obtain some idea about the probable shape of the interaction curve a somewhat easier problem, but having similar features, has been considered. This consists

Originally received 17th August 1953.
[The Aeronautical Quarterly, Vol. V, August 1954]

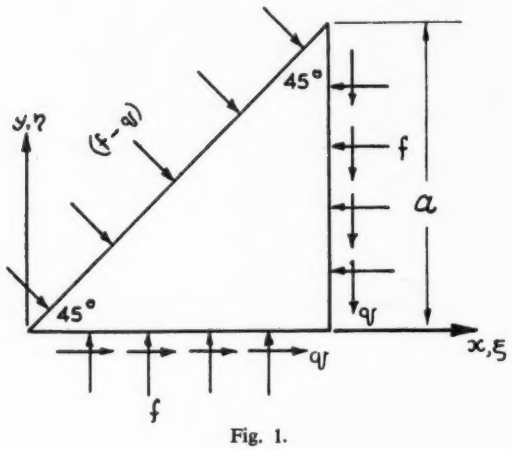

Fig. 1.

of a flat plate in the form of a right-angled isosceles triangle subjected to a combination of a shear stress q along the two perpendicular edges, together with a uniform compressive stress f (*i.e.* the same in all directions) as shown in Fig. 1. Four different sets of boundary conditions have been considered:—

(*i*) All edges simply-supported,

(*ii*) Perpendicular edges simply-supported, hypotenuse clamped,

(*iii*) Perpendicular edges clamped, hypotenuse simply-supported,

(*iv*) All edges clamped.

In each case both the plate and the applied loading are symmetrical about the bisector of the right angle of the triangle. The buckling mode is therefore either symmetrical or anti-symmetrical about this bisector, and it has been assumed that the symmetrical type of buckle gives the lowest critical stresses in each case*.

Sufficient numerical results were obtained to enable the interaction curve to be drawn for each of the first three sets of boundary conditions, as shown in Fig. 2. All three curves show similar effects; consequently no numerical calculations were performed on the plate with all edges clamped.

It will be seen from Fig. 2 that the two critical shear stresses are considerably different in each case, and that the critical compressive stress is appreciably increased by the application of a suitable amount of positive shear stress (*i.e.* in a direction tending to decrease the right angle of the triangle).

Although there is no mathematical analogy between the problem treated here and that of the oblique plate in combined compression and shear, nevertheless it is felt that it probably brings out the essential features of the interaction curve for an oblique plate.

*This was proved by numerical calculation to be true for the case of all edges simply-supported.

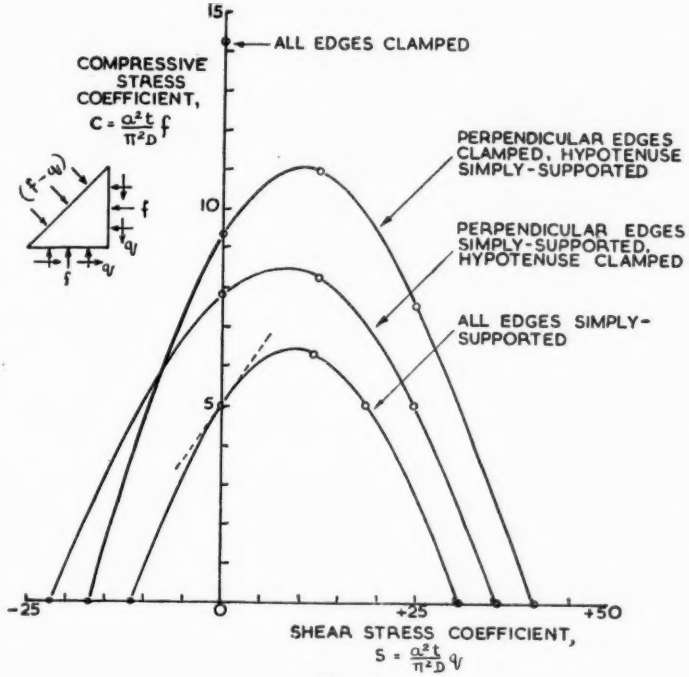

Fig. 2.

Notation

x, y	Cartesian co-ordinates (Fig. 1)
a	length of the equal sides of the triangle
ξ, η	$= x/a$ and y/a respectively
∇_1^2	$= \partial^2/\partial x^2 + \partial^2/\partial y^2$
∇^2	$= \partial^2/\partial \xi^2 + \partial^2/\partial \eta^2$
t	thickness of plate
E	Young's modulus
σ	Poisson's ratio
D	flexural rigidity $= Et^3/\{12(1-\sigma^2)\}$
w	lateral deflection of buckled plate
f	uniform compressive stress applied to the plate
q	shear stress applied to the plate
C, S	$= \frac{a^2t}{\pi^2 D} f, \frac{a^2t}{\pi^2 D} q$ respectively, buckling stress coefficients

m, n, p, q	integers
i, j, r, s	$=\frac{1}{2}(m-n), \frac{1}{2}(m+n), \frac{1}{2}(p-q), \frac{1}{2}(p+q)$ respectively
a_{mn}	coefficient of w_{mn} in the series for w
w_{mn}	one of a complete set of functions which satisfy the boundary conditions of the triangular plate
A_{mpnq}	general element of the stability determinant
F_m	function defined by equations (16)
$I_{mp}^{(\sigma)}$	definite integrals defined by equations (19)
δ_{mp}	$=1$ if $m=p$, $=0$ if $m \neq p$
Δ_{mp}	$=1$ if $(m+p)$ is even, $=0$ if $(m+p)$ is odd

2. Basic Theory

Consider a thin plate, of uniform thickness t and flexural rigidity D, in the shape of a right-angled isosceles triangle. Suppose that buckling has occurred under the action of a uniform compressive stress f in both the x and y directions and a uniform shear stress q, as shown in Fig. 1. The normal compressive stress acting on the hypotenuse is then equal to $(f-q)$, and there is no shear stress on this edge.

If w is the lateral deflection at the point (x, y) and if every edge is either clamped or simply-supported, the condition that the potential energy should be a minimum is given by the equation

$$\iint \left(D\nabla_1^4 w + ft\nabla_1^2 w + 2qt \frac{\partial^2 w}{\partial x \partial y} \right) \delta w \, dx dy = 0$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, δw is any small variation in w which satisfies the boundary conditions, and the integration extends over the area of the plate.

On transforming the variables x and y to the non-dimensional form ξ and η by the equations

$$\xi = x/a, \qquad \eta = y/a$$

where a is the length of the two equal sides of the triangle, the equation becomes

$$\int_0^1 \int_0^{\xi} \left(\frac{1}{\pi^2} \nabla^4 w + C\nabla^2 w + 2S \frac{\partial^2 w}{\partial \xi \partial \eta} \right) \delta w \, d\xi d\eta = 0 \, . \quad . \quad . \quad (1)$$

where

$$\nabla^2 = \partial^2/\partial \xi^2 + \partial^2/\partial \eta^2$$

and

$$C = \frac{a^2 t}{\pi^2 D} f, \qquad S = \frac{a^2 t}{\pi^2 D} q. \quad . \quad . \quad . \quad . \quad (2)$$

Suppose now that we represent the buckling mode by the series

$$w(\xi, \eta) = \sum_m \sum_n a_{mn} w_{mn}(\xi, \eta) \quad . \quad . \quad . \quad . \quad (3)$$

where the w_{mn} form a complete set of functions satisfying the boundary conditions of the plate. Then on substituting equation (3) into equation (1) and making

$$\delta w = \varepsilon w_{pq}(\xi, \eta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we obtain an infinite set of homogeneous simultaneous equations in the a_{mn}; thus

$$\sum_m \sum_n a_{mn} A_{mpnq} = 0 \text{ for all } p, q \quad . \quad . \quad . \quad . \quad (5)$$

where

$$A_{mpnq} = \int_0^1 \int_0^{\xi} \left(\frac{1}{\pi^2} \nabla^4 w_{mn} + C \nabla^2 w_{mn} + 2S \frac{\partial^2 w_{mn}}{\partial \xi \partial \eta} \right) w_{pq} \, d\xi d\eta. \quad . \quad . \quad . \quad (6)$$

A non-zero solution for the a_{mn} exists only if the determinant formed by the coefficients A_{mpnq} is zero. This gives a determinantal equation which must be satisfied by any critical combination of the buckling stress coefficients C and S.

The problem now is to find suitable expressions for the functions $w_{mn}(\xi, \eta)$ corresponding to each of the four sets of boundary conditions considered, and then to perform the integrations involved in equation (6) in order to obtain an expression for the general element of the stability determinant. Thereafter it is a matter of routine computation to obtain numerical results.

3. All Edges Simply-supported

To determine the functions w_{mn} in this case we notice first that the lateral deflection of a square plate, bounded by the lines $\xi = 0$, 1 and $\eta = 0$, 1, and simply-supported on all four edges can be represented quite generally by the series

$$w(\xi, \eta) = \sum_m \sum_n a_{mn} \sin m\pi\xi \sin n\pi\eta \quad . \quad . \quad . \quad (7)$$

which satisfies the boundary conditions term by term. If this is to represent the deflection of the triangular plate it is clear that w must be anti-symmetrical about the hypotenuse $\xi = \eta$,

i.e. $$w(\xi, \eta) + w(\eta, \xi) = 0.$$

Hence $$\sum_m \sum_n (a_{mn} + a_{nm}) \sin m\pi\xi \sin n\pi\eta = 0,$$

so that $$a_{nm} = -a_{mn}.$$

The series (7) now becomes

$$w(\xi, \eta) = \sum_m \sum_n a_{mn} (\sin m\pi\xi \sin n\pi\eta - \sin n\pi\xi \sin m\pi\eta).$$

Numerical calculations showed that for all combinations of compression and shear considered, the mode giving the smallest critical stresses is symmetrical about the bisector of the right angle of the triangle. It is easy to show that this requirement is satisfied if the series contains only those terms in which $(m+n)$ is odd. Also it is clear that interchanging m and n merely changes the sign of the general term of the series. Hence we have finally

$$w_{mn} = \sin m\pi\xi \sin n\pi\eta - \sin n\pi\xi \sin m\pi\eta \quad . \quad . \quad . \quad (8)$$

and we stipulate that m is even and n odd.

On substituting equation (8), together with a similar expression for w_{pq} (with p even and q odd), into equation (6), and performing the integration, we find that

$$\left.\begin{aligned} A_{mpnq} &= (m^2+n^2)(m^2+n^2-C) + \left(\frac{mn}{m^2-n^2}\right)^2 \frac{32S}{\pi^2} \text{ if } m=p \text{ and } n=q \\ &= -\frac{mnpq}{(m^2-q^2)(n^2-p^2)} \frac{32S}{\pi^2} \qquad \text{otherwise} \end{aligned}\right\} \quad . \quad . \quad (9)$$

Equation (8) is identical with that used by Klitchieff[3], who considered the problem of the buckling of a right-angled isosceles triangular plate with simply-supported edges under pure shear stress. It will be shown later, however, that the numerical value which he obtained for the critical shear stress is not accurate.

4. Perpendicular Edges Simply-supported, Hypotenuse Clamped

In this case we again start with equation (7), giving a general expression for the lateral deflection of a square plate which is simply-supported on all four edges. In order that the series should represent the deflection of the triangular plate, it is evident that w must be symmetrical about the diagonal $\xi=\eta$ and also that w must be zero along this diagonal.

From the condition of symmetry it follows that

$$w(\xi, \eta) - w(\eta, \xi) = 0.$$

Hence, on using equation (7)

$$\sum_m \sum_n (a_{mn} - a_{nm}) \sin m\pi\xi \sin n\pi\eta = 0,$$

so that $a_{nm} = a_{mn}$.

Equation (7) now becomes

$$w = \sum_m \sum_n a_{mn} (\sin m\pi\xi \sin n\pi\eta + \sin n\pi\xi \sin m\pi\eta) \quad . \quad . \quad (10)$$

where, in order to avoid repetition of terms, we stipulate that if $(m+n)$ is even we make $m \geqslant n$, and if $(m+n)$ is odd we make m even and n odd.

It will be assumed in the numerical calculations that for all combinations of shear and uniform compression the governing mode is symmetrical about the bisector of the right angle of the triangle, so that

$$w(\xi, \eta) = w(1-\eta, 1-\xi).$$

It is a simple matter to show that this requirement is satisfied if every term in the series in equation (10) is such that $(m+n)$ is even, and we therefore stipulate that $m \geqslant n$.

We still have to ensure that $w=0$ along the clamped edge of the triangle, *i.e.* along $\xi=\eta$. Equation (10) then gives

$$\sum_m \sum_n a_{mn} \sin m\pi\xi \sin n\pi\xi = 0,$$

so that

$$\sum_m \sum_n a_{mn} \cos(m+n)\pi\xi = \sum_m \sum_n a_{mn} \cos(m-n)\pi\xi.$$

On equating the coefficients of $\cos \lambda\pi\xi$ (where $\lambda = 0, 2, 4 \ldots$) in the series on the two sides of this equation, we obtain the following system of simultaneous equations:—

$$(\lambda=0) \quad a_{11} + a_{22} + a_{33} + a_{44} + \ldots\ldots = 0$$

$$(\lambda=2) \quad a_{31} + a_{42} + a_{53} + a_{64} + \ldots\ldots = a_{11}$$

$$(\lambda=4) \quad a_{51} + a_{62} + a_{73} + a_{84} + \ldots\ldots = a_{22} + a_{31}$$

$$(\lambda=6) \quad a_{71} + a_{82} + a_{93} + a_{10,4} + \ldots\ldots = a_{33} + a_{42} + a_{51}$$

$$(\lambda=8) \quad a_{91} + a_{10,2} + a_{11,3} + a_{12,4} + \ldots\ldots = a_{44} + a_{53} + a_{62} + a_{71}, \quad \text{etc.}$$

It will be seen that the first of these equations can be disregarded, since it can be obtained merely by adding all the remaining equations together. We therefore solve the remaining equations to obtain expressions for the coefficients a_{mm} in terms of the other coefficients of the type a_{mn} $(m \neq n)$. On substituting these expressions back into the series in equation (10) and collecting terms together we obtain finally the required expression for the deflection of the triangular plate; thus

$$w = \sum_m \sum_n a_{mn} w_{mn}$$

in which

$$w_{mn} = \sin m\pi\xi \sin n\pi\eta + \sin n\pi\xi \sin m\pi\eta + 2 \sin \tfrac{1}{2}(m-n)\pi\xi \sin \tfrac{1}{2}(m-n)\pi\eta - 2 \sin \tfrac{1}{2}(m+n)\pi\xi \sin \tfrac{1}{2}(m+n)\pi\eta \quad . \quad (11)$$

where $(m+n)$ is even and $m > n$.

It is a simple matter to check that the series satisfies term by term all the boundary conditions of the triangular plate.

On substituting equation (11), together with a similar expression for w_{pq} (with $(p+q)$ even and $p>q$), into equation (6), and performing the integration, the general element of the stability determinant becomes

$$A_{mpnq}=[(m^2+n^2)^2\,\delta_{mp}\delta_{nq}+8\,i^4\,(\delta_{ir}-\delta_{is})+8\,j^4\,(\delta_{js}-\delta_{jr})]-$$

$$-C[(m^2+n^2)\,\delta_{mp}\delta_{nq}+4\,i^2\,(\delta_{ir}-\delta_{is})+4\,j^2\,(\delta_{js}-\delta_{jr})]+$$

$$+\frac{32\,S}{\pi^2}\Bigg[\Delta_{mp}\left\{\frac{mnpq}{(m^2-p^2)(n^2-q^2)}+\frac{mnpq}{(m^2-q^2)(n^2-p^2)}\right\}+\Delta_{ip}\frac{2\,i^2pq}{(i^2-p^2)(i^2-q^2)}+$$

$$+\Delta_{rm}\frac{2\,r^2mn}{(r^2-m^2)(r^2-n^2)}-\Delta_{jp}\frac{2\,j^2pq}{(j^2-p^2)(j^2-q^2)}-\Delta_{sm}\frac{2\,s^2mn}{(s^2-m^2)(s^2-n^2)}+$$

$$+\Delta_{ir}\frac{2\,i^2r^2}{(i^2-r^2)^2}+\Delta_{js}\frac{2\,j^2s^2}{(j^2-s^2)^2}-\Delta_{is}\frac{2\,i^2s^2}{(i^2-s^2)^2}-\Delta_{jr}\frac{2\,j^2r^2}{(j^2-r^2)^2}\Bigg] \quad . \quad . \quad . \quad (12)$$

in which

$$\left.\begin{aligned} i&=\tfrac{1}{2}(m-n), \qquad j=\tfrac{1}{2}(m+n),\\ r&=\tfrac{1}{2}(p-q), \qquad s=\tfrac{1}{2}(p+q),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

Terms of the type δ_{mp} and Δ_{mp} are defined by

$$\left.\begin{aligned} \delta_{mp}&=1 \text{ if } m=p,\\ &=0 \text{ if } m\neq p,\\ \Delta_{mp}&=1 \text{ if } (m+p) \text{ is even},\\ &=0 \text{ if } (m+p) \text{ is odd}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

The numerical calculation of the elements of the stability determinant is not as arduous as it may appear from equation (12), since in every element a considerable number of terms are zero by virtue of the definitions of δ_{mp} and Δ_{mp}.

Because of the restrictions on m, n, p and q it is clear that the parameters i, j, r and s are all positive integers.

5. Perpendicular Edges Clamped, Hypotenuse Simply-supported

In this case we notice first that the lateral deflection of a square plate, bounded by the lines $\xi=0,1$ and $\eta=0,1$, which is clamped on all four edges can be represented quite generally by the series

$$w(\xi,\eta)=\sum_m\sum_n a_{mn}F_m(\xi)F_n(\eta) \quad . \quad . \quad . \quad (15)$$

where

$$\left.\begin{aligned} F_m(\xi)&=\sin\pi\xi\sin m\pi\xi\\ F_n(\eta)&=\sin\pi\eta\sin n\pi\eta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

and every term satisfies the clamped boundary conditions.

If this is to represent the deflection of the triangular plate, it is clear that w must be anti-symmetrical about the diagonal $\xi=\eta$. It follows that $a_{nm}=-a_{mn}$, and on assuming a mode which is symmetrical about the bisector of the right angle of the triangle, we obtain finally the expression

$$w_{mn}=F_m(\xi)F_n(\eta)-F_n(\xi)F_m(\eta) \quad . \quad . \quad . \quad (17)$$

in which m is even and n odd.

On substituting equation (17), together with a similar expression for w_{pq} (where p is even and q odd) into equation (6), the general element of the stability determinant becomes

$$A_{mpnq}=[I_{mp}^{(4)}I_{nq}^{(0)}+2I_{mp}^{(2)}I_{nq}^{(2)}+I_{mp}^{(0)}I_{nq}^{(4)}]+C[I_{mp}^{(2)}I_{nq}^{(0)}+I_{mp}^{(0)}I_{nq}^{(2)}]-\frac{32S}{\pi^2}I_{mq}^{(1)}I_{np}^{(1)} \quad . \quad . \quad . \quad (18)$$

where

$$\left.\begin{aligned} I_{mp}^{(0)}&=8\int_0^1 F_m(\xi)F_p(\xi)\,d\xi \\ I_{mp}^{(2)}&=\frac{8}{\pi^2}\int_0^1 F_m''(\xi)F_p(\xi)\,d\xi \\ I_{mp}^{(4)}&=\frac{8}{\pi^4}\int_0^1 F_m''''(\xi)F_p(\xi)\,d\xi \\ I_{mp}^{(1)}&=2\int_0^1 F_m'(\xi)F_p(\xi)\,d\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

and $F_m'(\xi)=(d/d\xi)F_m(\xi)$, etc.

The definite integrals defined by the equations (19) are given by

$I_{mp}^{(0)}=3$ if $m=p=1$

$=2$ if $m=p\neq 1$

$=-1$ if $m=p\pm 2$

$=0$ otherwise.

$I_{mp}^{(2)}=-4$ if $m=p=1$

$=-2(m^2+1)$ if $m=p\neq 1$

$=\frac{1}{4}(m+p)^2$ if $m=p\pm 2$

$=0$ otherwise.

$$I_{mp}^{(4)} = 16 \text{ if } m=p=1$$

$$= 2(m^4+6m^2+1) \text{ if } m=p\neq 1$$

$$= -\frac{1}{16}(m+p)^4 \text{ if } m=p\pm 2$$

$$= 0 \text{ otherwise.}$$

$$I_{mp}^{(1)} = \frac{16mp(m^2+p^2-2)}{(m^2-p^2)[(m^2-p^2)^2-8(m^2+p^2-2)]} \text{ if } (m+p) \text{ is odd}$$

$$= 0 \text{ otherwise.}$$

6. All Edges Clamped

In this final case we again start with equation (15), giving the deflection of a clamped square plate. The procedure is then exactly similar to that used in Section 4; thus we make w symmetrical about the diagonal $\xi=\eta$, and $w=0$ on this diagonal. We also assume that the buckling mode of the triangular plate is symmetrical about the bisector of the right angle. These requirements lead to the expression

$$w_{mn} = F_m(\xi)F_n(\eta) + F_n(\xi)F_m(\eta) + 2F_i(\xi)F_i(\eta) - 2F_j(\xi)F_j(\eta) \quad . \quad . \quad . \quad (20)$$

where $(m+n)$ is even and $m>n$, and i and j are defined by equation (13).

On substituting equation (20), together with a similar expression for w_{pq} (where $(p+q)$ is even, and $p>q$), into equation (6), the general element of the stability determinant becomes

$$A_{mpnq} = B_{mpnq} + CQ_{mpnq} + \frac{32S}{\pi^2}R_{mpnq} \quad . \quad . \quad . \quad (21)$$

where

$$\begin{aligned} B_{mpnq} = {} & [I_{mp}^{(4)}I_{nq}^{(0)} + 2I_{mp}^{(2)}I_{nq}^{(2)} + I_{mp}^{(0)}I_{nq}^{(4)}] + [I_{mq}^{(4)}I_{np}^{(0)} + 2I_{mq}^{(2)}I_{np}^{(2)} + I_{mq}^{(0)}I_{np}^{(4)}] + \\ & + 2[I_{mr}^{(4)}I_{nr}^{(0)} + 2I_{mr}^{(2)}I_{nr}^{(2)} + I_{mr}^{(0)}I_{nr}^{(4)}] + 2[I_{ip}^{(4)}I_{iq}^{(0)} + 2I_{ip}^{(2)}I_{iq}^{(2)} + I_{ip}^{(0)}I_{iq}^{(4)}] - \\ & - 2[I_{ms}^{(4)}I_{ns}^{(0)} + 2I_{ms}^{(2)}I_{ns}^{(2)} + I_{ms}^{(0)}I_{ns}^{(4)}] - 2[I_{jp}^{(4)}I_{jq}^{(0)} + 2I_{jp}^{(2)}I_{jq}^{(2)} + I_{jp}^{(0)}I_{jq}^{(4)}] + \\ & + 4[I_{ir}^{(4)}I_{ir}^{(0)} + (I_{ir}^{(2)})^2] + 4[I_{js}^{(4)}I_{js}^{(0)} + (I_{js}^{(2)})^2] - \\ & - 4[I_{is}^{(4)}I_{is}^{(0)} + (I_{is}^{(2)})^2] - 4[I_{jr}^{(4)}I_{ir}^{(0)} + (I_{jr}^{(2)})^2] \quad . \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

$$\begin{aligned} Q_{mpnq} = {} & [I_{mp}^{(2)}I_{nq}^{(0)} + I_{mp}^{(0)}I_{nq}^{(2)}] + [I_{mq}^{(2)}I_{np}^{(0)} + I_{mq}^{(0)}I_{np}^{(2)}] + \\ & + 2[I_{mr}^{(2)}I_{nr}^{(0)} + I_{mr}^{(0)}I_{nr}^{(2)}] + 2[I_{ip}^{(2)}I_{iq}^{(0)} + I_{ip}^{(0)}I_{iq}^{(2)}] - \\ & - 2[I_{ms}^{(2)}I_{ns}^{(0)} + I_{ms}^{(0)}I_{ns}^{(2)}] - 2[I_{jp}^{(2)}I_{jq}^{(0)} + I_{jp}^{(0)}I_{jq}^{(2)}] + \\ & + 4[I_{ir}^{(2)}I_{ir}^{(0)} + I_{js}^{(2)}I_{js}^{(0)} - I_{is}^{(2)}I_{is}^{(0)} - I_{jr}^{(2)}I_{jr}^{(0)}] \quad . \quad . \quad . \quad . \quad . \quad (23) \end{aligned}$$

$$R_{mpnq} = I_{mp}^{(1)} I_{nq}^{(1)} + I_{mq}^{(1)} I_{np}^{(1)} + 2 I_{mr}^{(1)} I_{nr}^{(1)} + 2 I_{ip}^{(1)} I_{iq}^{(1)} -$$

$$- 2 I_{ms}^{(1)} I_{ns}^{(1)} - 2 I_{jp}^{(1)} I_{jq}^{(1)} + 2 (I_{ir}^{(1)})^2 + 2 (I_{js}^{(1)})^2 -$$

$$- 2 (I_{is}^{(1)})^2 - 2 (I_{jr}^{(1)})^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In these expressions the definite integrals of the type $I_{mp}^{(q)}$ are as defined by equations (19). Many of them will be zero, but in spite of this the task of setting up the stability determinant is still very tedious, and in fact has not been done beyond the second order. Once the determinant has been set up numerically, there is no more work involved in the calculation of the roots than for any of the other three sets of boundary conditions.

7. Numerical Results

For each of the first three sets of boundary conditions sufficient numerical results were obtained to enable an interaction curve to be drawn covering the full range of possible combinations of shear and uniform compression.

The stability determinant is theoretically of infinite order but successive approximations to the roots can be obtained by including only a few appropriate rows and columns. As the order of the determinant increases the roots approach their exact values, the numerical magnitude being always too high. The roots were calculated by trial and error and the rows and columns which were included in the process of successive approximations were, in order, those corresponding to values of (m, n) and (p, q) given by the following sequences.

(*i*) All edges simply-supported.
(2, 1) (2, 3) (4, 1) (2, 5) (4, 3) (6, 1) (2, 7) (4, 5) (6, 3) (8, 1) (6, 5)

(*ii*) Perpendicular edges simply-supported, hypotenuse clamped.
(3, 1) (4, 2) (5, 1) (5, 3) (6, 2) (7, 1) (6, 4) (7, 3) (8, 2) (9, 1)

(*iii*) Perpendicular edges clamped, hypotenuse simply-supported.
The same as (*i*).

The numerical results are shown in Tables I, II and III, which also give the order of the determinant necessary to obtain sufficient accuracy, as indicated by the successive values of the roots as the order of the determinant increases.

It will be seen that for the plate with all edges simply-supported the solution for pure compression ($C=5$, $S=0$) is obtained from a first order determinant. In fact this result is exact, for on substituting $S=0$ into equations (9) it is clear that every element of the stability determinant vanishes except those on the principal diagonal which have the value

$$(m^2+n^2)(m^2+n^2-C).$$

The value of the determinant is then equal to the product of the diagonal elements and one of these must vanish, giving $C=m^2+n^2$, where m and n must be chosen

TABLE I

ALL EDGES SIMPLY-SUPPORTED

Compressive stress coefficient C	0	5	6·29	5·01	0
Shear stress coefficient S	− 11·55	0	11·57	18·51	30·89
Order of determinant	8th	1st	9th	9th	11th

TABLE II

PERPENDICULAR EDGES SIMPLY-SUPPORTED, HYPOTENUSE CLAMPED

Compressive stress coefficient C	0	7·82	8·23	5·02	0
Shear stress coefficient S	− 22·02	0	12·34	24·67	35·53
Order of determinant	8th	4th	8th	10th	10th

TABLE III

PERPENDICULAR EDGES CLAMPED, HYPOTENUSE SIMPLY-SUPPORTED

Compressive stress coefficient C	0	9·35	10·98	7·55	0
Shear stress coefficient S	− 17·12	0	12·34	24·67	40·21
Order of determinant	8th	5th	8th	10th	10th

to give the smallest possible value of C. Since m is even and n odd, it follows that $m=2$ and $n=1$, so that $C=5$. The buckling mode then has the simple form

$$w=\sin 2\pi\xi \sin \pi\eta - \sin \pi\xi \sin 2\pi\eta.$$

A further result for the simply-supported plate, not shown in Table I, relates to the value of dC/dS at the point $C=5$, $S=0$. If we differentiate the stability determinant with respect to S and then put $C=5$, $S=0$, it follows immediately that at this point

$$\frac{dC}{dS}=\frac{128}{45\pi^2}=0{\cdot}288. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The results given in Tables I, II and III are shown plotted in Fig. 2, the broken line representing the tangent defined by equation (25). For all three sets of boundary conditions it will be seen that the two (positive and negative) critical shear stresses are considerably different, and that an appreciable increase in critical compressive stress is produced by the application of a suitable amount of positive shear.

Because of the complexity of equations (21) to (24), the interaction curve for a plate with all edges clamped has not been derived. However, the point

corresponding to pure compression (*i.e.* $S=0$) has been calculated approximately, using a second order determinant, and is plotted in Fig. 2. The terms included in the series for w were those for which $m=3$, $n=1$ and $m=4$, $n=2$. Using the first term only gives $C=14{\cdot}29$, while both terms together give $C=14{\cdot}24$. The closeness of these two values, and the experience with the other three sets of boundary conditions, suggest that this latter value is probably accurate to within one per cent.

The problem of the buckling of a simply-supported triangular plate under pure shear has previously been considered by Klitchieff[3]. He calculated only the positive critical shear stress and obtained the value $S=+46{\cdot}32$, as compared with $S=+30{\cdot}89$ given in Table I. The reason for this discrepancy lies in the fact that Klitchieff omitted several terms from the series for w which make a considerable contribution to the mode and he therefore obtained a value which is too high by about 50 per cent.

Acknowledgment

The author wishes to acknowledge, with thanks, the assistance of Mrs. Lurline Poulton in evaluating the determinants.

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Subsonic Compressible Flow Past Bluff Bodies

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Summary: In this paper the Janzen[1]-Rayleigh[2] method is used to calculate the velocity potential for the steady subsonic flow of a compressible, inviscid fluid past a prolate spheroid. The fluid velocity at a point on the body is calculated. The analytic form obtained for this velocity differs, from that giving the velocity which an incompressible fluid would possess at the same point on the body, by a correction factor. The factor is an infinite series of first derivatives of Legendre functions of the first kind and odd order. The first three coefficients in this series are computed for bodies of certain axis ratios, and graphs of the values of these coefficients against axis ratio are plotted. The behaviour of the n^{th} coefficient for large values of n is given. Results for slender ellipsoids, considering these as a limiting case of the family of ellipsoids just referred to, are obtained and are found to agree with the usual slender-body theory. Using these an attempt is made to continue the graphs of the first three coefficients in the correction factor series for the whole range of axis ratios of the ellipsoids in the system, namely zero to unity. The results obtained for the bluff-nosed ellipsoids may be used to estimate the effects of compressibility on the pressure distribution over the front of a general bluff-nosed body in steady flow.

1. Introduction

The Janzen[1]-Rayleigh[2] method of calculating subsonic flows past obstacles by expanding the velocity potential in powers of the Mach number has been extensively applied to two-dimensional problems[3]. It has also been applied to three-dimensional flow by (*a*) Rayleigh[2], (*b*) Lamla[4], (*c*) Schmieden[5] and (*d*) Sauer[6]. Rayleigh calculated the velocity potential for subsonic flow past a sphere and obtained a solution of the form $\phi = \phi_0 + M^2\phi_1$. Lamla has extended this work and has found a value for ϕ including two further terms, those in M^4 and M^6. Schmieden has found the first two terms of the stream function for subsonic flow past an ellipsoid of revolution and has also calculated the velocity at the position of maximum thickness of the body and the critical Mach number of the flow. Sauer gives a numerical method which can be applied to calculate any axially symmetric three-dimensional flow.

Our knowledge of the compressibility effects for flow past bodies of revolution is, therefore, limited to (*i*) the work already mentioned and (*ii*) slender bodies for which the usual perturbation theory can be applied. The present paper is an attempt to bridge the gap and in it the Janzen-Rayleigh method is applied to give some indication of the compressibility effects in uniform flow past bluff bodies of revolution. To formulate the problem mathematically a prolate spheroid is considered as a typical bluff body and the flow past it is in a direction parallel to its

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Originally received December 1952.

[The Aeronautical Quarterly, Vol. V, July 1954]

polar radius. By varying the eccentricity of the meridian section of this spheroid, nose shapes of varying curvatures are obtained. Thus if the nose region of any bluff body, for example an airship or an engine nacelle, can be approximated to in shape by a prolate spheroid, then the flow in that region of such a body can be estimated using the results given here.

Only the first two terms in the expansion of the velocity potential in powers of the free stream Mach number M are calculated, the first term being the incompressible value of the velocity potential and the second the term in M^2. However, for low values of M the neglect of M^4 and higher powers of M compared with M^2 gives a reasonable approximation. For flow at high Reynolds numbers the neglect of viscosity will lead to a solution which gives a reasonable picture of the flow over the front portion of the body, for here the actual flow approximates fairly closely to that calculated from potential theory. In practice the effects of viscosity are manifest mainly towards the rear of the body where the boundary layer begins to thicken and finally separate, thus producing the wake. Hence, on the rear of the body the solution obtained by neglecting viscosity cannot be expected to give any useful predictions as in this region the viscous effects are the dominant feature of the flow. However, the separation at the rear is known not to affect greatly the pressure distribution over the front of the body and in view of this fact the solution found in this paper may be expected to give a reasonably accurate representation of the compressibility effects in the front of the body.

In the application of the Janzen-Rayleigh method to this particular problem the equation satisfied by the velocity potential ϕ in steady subsonic flow, namely

$$\nabla^2\phi = \frac{1}{2a^2}\operatorname{grad}\phi \operatorname{grad} q^2, \qquad (1)$$

(where a is the speed of sound through the fluid at a point where the density of the fluid is ρ and its speed is q) is expressed in ellipsoidal co-ordinates (ξ, μ, ω). The fluid is assumed to obey the adiabatic law, and hence

$$2a^2 + (\gamma - 1)q^2 = 2a_\infty{}^2 + (\gamma - 1)U^2, \qquad (2)$$

where U is the free stream velocity, a_∞ is the speed of sound through fluid moving with this velocity and of density ρ_∞, and γ is the ratio of the specific heat of the fluid at constant pressure to that at constant volume. For the flow considered, the free stream velocity is U in the negative Ox-direction and parallel to the polar axis of the ellipsoid and hence M, the free stream Mach number, is given by $M = U/a_\infty$. If now, a series expansion for ϕ in terms of M is assumed, then

$$\phi = \sum_{n=0}^{n=\infty} \phi_n M^{2n}, \qquad (3)$$

where ϕ_0 is the value of ϕ when $M = 0$, that is the solution for the incompressible flow of a stream with velocity U past the given body. In this paper, only ϕ_1 is calculated so that the value of ϕ which is found has the form

$$\phi = \phi_0 + M^2\phi_1. \qquad (4)$$

The expression obtained for ϕ_1 is of the form

$$\phi_1 = \sum_{n=0}^{n=\infty} b_{2n+1} P_{2n+1}(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the coefficients b_{2n+1} are independent of μ, and $P_{2n+1}(\mu)$ is the Legendre coefficient of the first kind. The fluid velocity at a point on the body is then calculated and it is expressed in terms of the velocity which the fluid would possess if it were supposed incompressible. These results, together with some computations, are described and discussed fully in Section 6.

Notation

A constant; *see* equation (8)

A.R. axis ratio

$(C_p)_0 = (p - p_\infty)/(\frac{1}{2}\rho_\infty U^2)$, pressure coefficient for incompressible flow

$(C_p)_{0,\beta(1-M^2)^{\frac{1}{2}}/\alpha}$ pressure coefficient at a point on a body of thickness ratio $\beta(1-M^2)^{\frac{1}{2}}/\alpha$ in incompressible flow

$(C_p)_{M,\beta/\alpha}$ pressure coefficient at a point on a body of thickness ratio β/α in compressible flow

$F(\mu) = t^2/2$ in case of slender bodies; *see* equations (49) and (50)

$= \sum_{n=0}^{n=\infty} [H_{2n+1}(\zeta_0) P_{2n+1}'(\mu)]$ in case of bluff ellipsoids; *see* equation (29)

$F_0(\mu) = F(\mu)$ in case of slender bodies

$= \sum_{n=0}^{n=2} [H_{2n+1}(\zeta_0) P_{2n+1}'(\mu)]$ in case of bluff ellipsoids

$F_1(\zeta), F_2(\zeta), F_3(\zeta)$ *see* equation (11)

Φ_1 *see* equation (12)

$G_{2n+1}(\zeta)$ *see* equations (20) and (22)

$H_{2n+1}(\zeta_0)$ *see* equation (28)

$M = U/a_\infty$, free stream Mach number

$P_{2n+1}(\mu)$ Legendre coefficient of first kind

$P_{2n+1}'(\mu) = \dfrac{d}{d\mu}[P_{2n+1}(\mu)]$

$Q_{2n+1}(\zeta)$ Legendre coefficient of second kind

$Q_{2n+1}'(\zeta) = \dfrac{d}{d\zeta}[Q_{2n+1}(\zeta)]$

U free stream velocity

a speed of sound at any point in fluid

a_∞ speed of sound in free stream

α polar radius of spheroid

b_{2n+1} *see* equation (5)

β equatorial radius of spheroid

c_n constant; *see* equation (24)

c_{2n+1}' constant

γ ratio of specific heats of fluid at constant pressure and constant volume

d_n', d_{2n+1} constants

$\delta = \zeta_0 - 1$; *see* Section 5

e eccentricity of meridian section of spheroid

$f_1(\zeta), f_2(\zeta)$ *see* equations (12), (13), (14) and (16)

ϕ velocity potential for compressible flow past the spheroid

ϕ_0 velocity potential for incompressible flow past the spheroid

$\phi_1, \phi_2, \ldots$ correction terms to be added to ϕ_0 to obtain ϕ

ϕ_p perturbation potential; *see* Section 5

$g_{2n+1}(\zeta)$ *see* equation (26)

$K = \alpha/\zeta_0$ constant

λ angle between the tangent to the meridian section of the ellipsoid and the x-axis

p static pressure at any point in the fluid

p_∞ static pressure in free stream

q speed of fluid at any point in compressible flow

$\mathbf{q}$ velocity of fluid at any point in compressible flow

q_0 speed of fluid at any point in incompressible flow

$\mathbf{q}_0$ velocity of fluid at any point in incompressible flow

$q_{b,0}$ speed of fluid at point on body surface in incompressible flow

$q_{b,M}$ speed of fluid at point on body surface in compressible flow

r_0 radius of section of spheroid perpendicular to x-axis

ρ density of fluid at any point

ρ_∞ density of fluid in free stream

s length of arc of a meridian section of the body measured from the forward stagnation point; *see* equation (52)

$t = \beta/\alpha$, thickness ratio

u_1, v_1 components of perturbation velocity; *see* Section 5

(x, y, z) Cartesian co-ordinates

(ζ, μ, ω) ellipsoidal co-ordinates; *see* equation (6) for relation between these and (x, y, z)

ζ_0 a particular value of ζ, defining a prolate spheroid

2. Derivation of the Velocity Potential of the Flow

Consider a prolate spheroid with polar and equatorial radii α and β respectively. Choose rectangular Cartesian axes $Oxyz$ at O, the centre of the spheroid, and fixed in space, and let the polar radius coincide with Ox. Suppose a steady stream with velocity U in the negative Ox-direction flows past the body. With co-ordinates

$$x=K\mu\zeta, \quad y=K(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega, \quad z=K(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega, \tag{6}$$

where K is a constant and (x, y, z) are the Cartesian co-ordinates of a point referred to the axes $Oxyz$, then this point may also be denoted by (ζ, μ, ω), the so-called ellipsoidal co-ordinates. These latter form an orthogonal system and in particular $\zeta=\zeta_0$, a constant, denotes an ellipsoid of revolution with $\alpha=K\zeta_0$ and $\beta=K(\zeta_0^2-1)^{\frac{1}{2}}$. In terms of the co-ordinates (ζ, μ, ω) the expression obtained for ϕ_0, the velocity potential for the motion of a steady stream of incompressible fluid with velocity U in the negative Ox-direction past the ellipsoid of revolution $\zeta=\zeta_0$, is given by[7]

$$\phi_0=-UK\mu\zeta-A\mu Q_1(\zeta), \quad . \quad . \quad . \quad . \quad . \tag{7}$$

where the fluid velocity $\mathbf{q}_0=\operatorname{grad}\phi_0$ (the flow being irrotational), $Q_1(\zeta)$ is the Legendre coefficient of the second kind and A is a constant, namely

$$A=U\alpha\Big/\left\{\frac{1}{1-e^2}-\frac{1}{2e}\log\frac{1+e}{1-e}\right\} \quad . \quad . \quad . \quad . \tag{8}$$

where e is the eccentricity of the meridian section of the ellipsoid $(=1/\zeta_0)$.

For a compressible fluid in steady motion the flow is irrotational and hence a velocity potential ϕ exists. ϕ satisfies the equation (1)

$$\nabla^2\phi=\frac{1}{2a^2}\operatorname{grad}\phi\operatorname{grad}q^2,$$

and since in ellipsoidal co-ordinates the operator ∇^2 is given by

$$\nabla^2\equiv\frac{1}{K^2(\zeta^2-\mu^2)}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right]+\frac{\partial}{\partial\zeta}\left[(\zeta^2-1)\frac{\partial}{\partial\zeta}\right]+\frac{\partial}{\partial\omega}\left[\frac{(\zeta^2-\mu^2)}{(1-\mu^2)(\zeta^2-1)}\frac{\partial}{\partial\omega}\right]\right\},$$

and here the motion is symmetrical about Ox, and thus ϕ is independent of ω, then equation (1) becomes

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{\partial\phi}{\partial\zeta}\right\}=\frac{(1-\mu^2)}{2a^2}\frac{\partial\phi}{\partial\mu}\frac{\partial q^2}{\partial\mu}+\frac{(\zeta^2-1)}{2a^2}\frac{\partial\phi}{\partial\zeta}\frac{\partial q^2}{\partial\zeta}. \tag{9}$$

The boundary conditions for the flow are

(*i*) the normal fluid velocity on the surface of the ellipsoid is zero, that is $\partial\phi/\partial\zeta=0$ when $\zeta=\zeta_0$ for all values of μ, and

(*ii*) at large distances ϕ tends to the value of the velocity potential for a free stream flowing with velocity U parallel to the negative Ox-direction, that is $\phi\longrightarrow-UK\mu\zeta$ for large values of ζ.

Assume a solution of equation (9) of the form $\phi=\phi_0+M^2\phi_1$, where $M(=U/a_\infty)$ is the free stream Mach number. From equation (2)

$$2a^2+(\gamma-1)\,q^2=2a^2{}_\infty+(\gamma-1)\,U^2,$$

it follows that $$\frac{1}{a^2}=\frac{M^2}{U^2}\left[1-\frac{\gamma-1}{2}M^2\left(1-\frac{q^2}{U^2}\right)+O(M^4)\right],$$

and hence, if in equation (9) the expressions for $1/a^2$ and ϕ are substituted, then equating coefficients of M^2 gives

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\phi_1}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{\partial\phi_1}{\partial\zeta}\right\}=\frac{(1-\mu^2)}{2U^2}\frac{\partial\phi_0}{\partial\mu}\frac{\partial q_0{}^2}{\partial\mu}+\frac{(\zeta^2-1)}{2U^2}\frac{\partial\phi_0}{\partial\zeta}\frac{\partial q_0{}^2}{\partial\zeta}, \quad (10)$$

since $\nabla^2\phi_0=0$, where q_0 is calculated using the velocity potential ϕ_0. Evaluating the right hand side of equation (10) gives

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\phi_1}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{\partial\phi_1}{\partial\zeta}\right\}=\frac{A}{U^2K^2(\zeta^2-\mu^2)^2}[\mu F_1(\zeta)+\mu^3F_3(\zeta)+\mu^5F_5(\zeta)], \quad (11)$$

where $$F_1(\zeta)=3\{UK\zeta+AQ_1(\zeta)\}^2-\frac{2A}{(\zeta^2-1)}\{UK\zeta+AQ_1(\zeta)\},$$

$$F_3(\zeta)=\frac{1-5\zeta^2}{\zeta^2}\{UK\zeta+AQ_1(\zeta)\}^2-\frac{2A(1-3\zeta^2)}{\zeta^2(\zeta^2-1)}\{UK\zeta+AQ_1(\zeta)\}+\frac{A^2(1-2\zeta^2)}{\zeta^2(\zeta^2-1)^2}$$

and $$F_5(\zeta)=\frac{1}{\zeta^2}\{UK\zeta+AQ_1(\zeta)\}^2-\frac{2A}{\zeta^2(\zeta^2-1)}\{UK\zeta+AQ_1(\zeta)\}+\frac{A^2}{\zeta^2(\zeta^2-1)^2}.$$

Write $$\phi=\frac{\Phi_1}{M^2}+\frac{A\mu}{U^2K^2}f_1(\zeta)+\frac{A\mu}{U^2K^2(\zeta^2-\mu^2)}f_2(\zeta) \quad (12)$$

where $f_1(\zeta)$ and $f_2(\zeta)$ are functions to be determined and substitute in equation (11). If $f_1(\zeta)$ is chosen to be a solution of the equation

$$(\zeta^2-1)\frac{d^2f_1}{d\zeta^2}+2\zeta\frac{df_1}{d\zeta}-2f_1=\frac{1}{\zeta^2}\left[UK\zeta+AQ_1(\zeta)-\frac{A}{\zeta^2-1}\right]^2, \quad (13)$$

and $f_2(\zeta)$ is chosen as

$$f_2(\zeta)=\frac{1}{2}\{UK\zeta+AQ_1(\zeta)\}^2+\frac{A^2}{4}Q_0{}^2(\zeta)-\frac{A^2}{4(\zeta^2-1)}, \quad (14)$$

then the equation for Φ_1 becomes

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Phi_1}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{\partial\Phi_1}{\partial\zeta}\right\}=\frac{A^3\mu}{a_\infty{}^2K^2(\zeta^2-\mu^2)^2}\left[\frac{1}{(\zeta^2-1)}-2Q_1(\zeta)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From equation (13),

$$f_1(\zeta)=P_1(\zeta)\int_{\infty}^{\zeta}Q_1(\xi)[UK+AQ_1'(\xi)]^2\,d\xi-Q_1(\zeta)\int_{\zeta_0}^{\zeta}P_1(\xi)[UK+AQ_1'(\xi)]^2\,d\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

To solve equation (15) consider the expansion

$$\frac{\mu}{\zeta^2-\mu^2}=\sum_{n=0}^{n=\infty}(4n+3)P_{2n+1}(\mu)Q_{2n+1}(\zeta), \quad . \quad . \quad . \quad (17)$$

which follows from the expansion given in Ref. 8. For this problem the expansion is certainly valid, as here ζ is real and greater than unity for all surfaces of the family except the degenerate case, which is of no interest, and also μ lies between -1 and $+1$. Differentiating (17) with respect to ζ,

$$\frac{\mu}{(\zeta^2-\mu^2)^2}=-\sum_{n=0}^{n=\infty}\frac{(4n+3)}{2\zeta}Q_{2n+1}'(\zeta)P_{2n+1}(\mu). \quad . \quad . \quad . \quad (18)$$

Hence substituting in equation (15),

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Phi_1}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{\partial\Phi_1}{\partial\zeta}\right\}=\frac{A^3}{a_\infty{}^2K^2}\sum_{0=u}^{n=\infty}\frac{4n+3}{2}Q_{2n+1}'(\zeta)\left[\frac{2Q_1(\zeta)}{\zeta}-\frac{1}{\zeta(\zeta^2-1)}\right]P_{2n+1}(\mu). \quad (19)$$

Assume a solution for Φ_1 of the form

$$\Phi_1=\frac{A^3}{a_\infty{}^2K^2}\sum_{n=0}^{n=\infty}\frac{4n+3}{2}G_{2n+1}(\zeta)P_{2n+1}(\mu), \quad . \quad . \quad . \quad (20)$$

where $G_{2n+1}(\zeta)$ are functions to be determined, and substitute this series in equation (19). Equating coefficients of $P_{2n+1}(\mu)$, we then obtain

$$\frac{\partial}{\partial\zeta}\left\{(\zeta^2-1)\frac{dG_{2n+1}(\zeta)}{d\zeta}\right\}-(2n+1)(2n+2)G_{2n+1}(\zeta)=\left[\frac{2Q_1(\zeta)Q_{2n+1}'(\zeta)}{\zeta}-\frac{Q_{2n+1}'(\zeta)}{\zeta(\zeta^2-1)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence

$$G_{2n+1}(\zeta)=P_{2n+1}(\zeta)\int_{\infty}^{\zeta}Q_{2n+1}(\xi)\left\{\frac{2Q_1(\xi)Q_{2n+1}'(\xi)}{\xi}-\frac{Q_{2n+1}'(\xi)}{\xi(\xi^2-1)}\right\}d\xi-$$
$$-Q_{2n+1}(\zeta)\int_{\zeta_0}^{\zeta}P_{2n+1}(\xi)\left\{\frac{2Q_1(\xi)Q_{2n+1}'(\xi)}{\xi}-\frac{Q_{2n+1}'(\xi)}{\xi(\xi^2-1)}\right\}d\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

this being true for $n=0, 1, 2, \ldots$

As a particular integral for equation (11), we then have

$$\phi_1 = \frac{A}{U^2K^2}\left[\mu f_1(\zeta) + \frac{\mu}{(\zeta^2-\mu^2)} f_2(\zeta) + A^2 \sum_{n=0}^{n=\infty} \frac{4n+3}{2} G_{2n+1}(\zeta) P_{2n+1}(\mu)\right], \quad (23)$$

where the functions $f_1(\zeta)$, $f_2(\zeta)$ and $G_{2n+1}(\zeta)$ $(n=0, 1, 2, \ldots)$ are given by equations (16), (14) and (22) respectively. The general solution of equation (10) can then be written as

$$\phi = \phi_0 + \frac{A}{a_\infty{}^2K^2}\left[\mu f_1(\zeta) + \frac{\mu}{(\zeta^2-\mu^2)} f_2(\zeta) + A^2 \sum_{n=0}^{n=\infty} \frac{4n+3}{2} G_{2n+1}(\zeta) P_{2n+1}(\mu)\right] + \\ + \sum_{n=0}^{n=\infty} c_n P_n(\mu) P_n(\zeta) + \sum_{n=0}^{n=\infty} d_n' P_n(\mu) Q_n(\zeta), \quad . \quad . \quad . \quad (24)$$

the last two terms being the complementary function for equation (9) and c_n, d_n' being constants to be determined by applying the boundary conditions (*i*) and (*ii*). Now ϕ_0, being the solution for incompressible flow, will satisfy both (*i*) and (*ii*) and thus ϕ_1 must also separately satisfy (*i*) and must tend to zero at large distances from the ellipsoid, that is as ζ tends to infinity. Hence, from the latter condition

$$\sum_{n=0}^{n=\infty} c_n P_n(\mu) P_n(\zeta) + \sum_{n=0}^{n=\infty} d_n' P_n(\mu) Q_n(\zeta)$$

must tend to zero as ζ tends to infinity, since all the terms of the particular integral do so. This can only be true if $c_n = 0$ for all n. The constants d_n' still remain to be determined, and for this purpose (*i*) must be used. We have

$$\frac{A}{a_\infty{}^2K^2}\left\{\frac{\partial}{\partial\zeta}\left[P_1(\mu) f_1(\zeta) + f_2(\zeta) \sum_{n=0}^{n=\infty} (4n+3) P_{2n+1}(\mu) Q_{2n+1}(\zeta) + \right.\right. \\ \left.\left. + \frac{A^2}{2} \sum_{n=0}^{n=\infty} (4n+3) G_{2n+1}(\zeta) P_{2n+1}(\mu)\right]\right\}_{\zeta=\zeta_0} + \sum_{n=1}^{n=\infty} d_n' P_n(\mu)\left[\frac{dQ_n(\zeta)}{d\zeta}\right]_{\zeta=\zeta_0} = 0.$$

Equating coefficients of $P_n(\mu)$, then $d_n' = 0$ for all even values of n. Writing

$$d_{2n+1}' = \frac{A}{a_\infty{}^2K^2} \frac{4n+3}{2} d_{2n+1}$$

then we have, for $n = 0, 1, 2 \ldots .$,

$$\left\{\frac{d}{\partial\zeta}\left[P_1(\mu) f_1(\zeta) + f_2(\zeta) \sum_{n=0}^{n=\infty} (4n+3) P_{2n+1}(\mu) Q_{2n+1}(\zeta) + \right.\right. \\ \left.\left. + A^2 \sum_{n=0}^{n=\infty} \frac{4n+3}{2} G_{2n+1}(\zeta) P_{2n+1}(\mu)\right]\right\}_{\zeta=\zeta_0} + \sum_{n=0}^{n=\infty} \frac{4n+3}{2} d_{2n+1}\left[\frac{d}{d\zeta} Q_{2n+1}'(\zeta)\right]_{\zeta=\zeta_0} P_{2n+1}(\mu) = 0.$$

Hence

$$\left.\begin{aligned} &\left[\frac{df_1}{d\zeta}\right]_{\zeta=\zeta_0} + 3\left[\frac{d}{d\zeta}\{Q_1(\zeta) f_2(\zeta)\}\right]_{\zeta=\zeta_0} + \frac{3A^2}{2}\left[\frac{dG_1}{d\zeta}\right]_{\zeta=\zeta_0} + \frac{3}{2} d_1\left[\frac{dQ_1(\zeta)}{d\zeta}\right]_{\zeta=\zeta_0} = 0 \\ &\text{and} \\ &\left[\frac{d}{d\zeta}\{f_2(\zeta) Q_{2n+1}(\zeta)\}\right]_{\zeta=\zeta_0} + \frac{A^2}{2}\left[\frac{dG_{2n+1}(\zeta)}{d\zeta}\right]_{\zeta=\zeta_0} + \frac{1}{2} d_{2n+1}\left[\frac{dG_{2n+1}(\zeta)}{d\zeta}\right]_{\zeta=\zeta_0} = 0 \end{aligned}\right\} \quad (25)$$

for $n=1, 2, 3 \ldots$. These relations determine the constants d_{2n+1} and hence ϕ is determined. The form of the solution is

$$\phi=\phi_0+\frac{A}{a_\infty{}^2K^2}f_1(\zeta)P_1(\mu)+\frac{A}{a_\infty{}^2K^2}\sum_{n=0}^{n=\infty}(4n+3)P_{2n+1}(\mu)\times$$
$$\times\left\{f_2(\zeta)Q_{2n+1}(\zeta)+\frac{A^2}{2}G_{2n+1}(\zeta)+\frac{d_{2n+1}}{2}Q_{2n+1}(\zeta)\right\}$$
$$=\sum_{n=0}^{n=\infty}c_{2n+1}{}'g_{2n+1}(\zeta)P_{2n+1}(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where $c_{2n+1}{}'$ are constants and $g_{2n+1}(\zeta)$ are known functions of ζ.

3. Calculation of the Fluid Velocity at a Point on the Body

To obtain the fluid velocity at a point on the body only $(\partial\phi/\partial\mu)_{\zeta=\zeta_0}$ is required, since $(\partial\phi/\partial\zeta)_{\zeta=\zeta_0}$ is zero from the boundary condition (*i*). From equation (26)

$$\frac{\partial\phi}{\partial\mu}=\frac{\partial\phi_0}{\partial\mu}+\frac{A}{a_\infty{}^2K^2}f_1(\zeta)+\frac{A}{a_\infty{}^2K^2}\sum_{n=0}^{n=\infty}(4n+3)\frac{dP_{2n+1}(\mu)}{d\mu}\left[f_2(\xi)Q_{2n+1}(\zeta)+\right.$$
$$\left.+\frac{A^2}{2}G_{2n+1}(\zeta)+\frac{d_{2n+1}}{2}Q_{2n+1}(\zeta)\right].$$

Hence

$$\left(\frac{\partial\phi}{\partial\mu}\right)_{\zeta=\zeta_0}=\left(\frac{\partial\phi_0}{\partial\mu}\right)_{\zeta=\zeta_0}+\frac{A}{a_\infty{}^2K^2}f_1(\zeta_0)+\frac{A^3}{2a_\infty{}^2K^2}\sum_{n=0}^{n=\infty}(4n+3)\frac{d}{d\mu}P_{2n+1}(\mu)P_{2n+1}(\zeta_0)\times$$
$$\times\int_\infty^{\zeta_0}Q_{2n+1}(\xi)\frac{d}{d\xi}Q_{2n+1}(\xi)\left[\frac{2Q_1(\xi)}{\xi}-\frac{1}{\xi(\xi^2-1)}\right]d\xi+\frac{3A}{a_\infty{}^2K^2}Q_1(\zeta_0)\left[f_2(\zeta_0)+\frac{d_1}{2}\right]+$$
$$+\frac{A}{a_\infty{}^2K^2}\sum_{n=1}^{n=\infty}(4n+3)\frac{dP_{2n+1}(\mu)}{d\mu}Q_{2n+1}(\zeta_0)\left[f_2(\zeta_0)+\frac{d_{2n+1}}{2}\right],$$

having substituted for $G_{2n+1}(\zeta_0)$ from equation (22). Simplifying the last expression still further by substituting for $f_1(\zeta_0)$ and $f_2(\zeta_0)$, then

$$\left(\frac{\partial\phi}{\partial\mu}\right)_{\zeta=\zeta_0}=\left(\frac{\partial\phi_0}{\partial\mu}\right)_{\zeta=\zeta_0}-\frac{A}{a_\infty{}^2K^2(\zeta_0{}^2-1)Q_1{}'(\zeta_0)}\int_\infty^{\zeta_0}Q_1(\xi)[UK+AQ_1{}'(\xi)]^2d\xi+$$
$$+\frac{A}{a_\infty{}^2K^2}\sum_{n=0}^{n=\infty}\frac{4n+3}{2}P_{2n+1}{}'(\mu)\left\{\frac{Q_{2n+1}{}^2(\zeta_0)U^2K^2}{Q_1{}'(\zeta_0)Q_{2n+1}{}'(\zeta_0)(\zeta_0{}^2-1)}-\right.$$
$$\left.-\frac{A^2}{(\zeta_0{}^2-1)Q_{2n+1}{}'(\zeta_0)}\int_\infty^{\zeta_0}Q_{2n+1}(\xi)Q_{2n+1}{}'(\xi)\left[\frac{2Q_1(\xi)}{\xi}-\frac{1}{\xi(\xi^2-1)}\right]d\xi\right\}.$$

But $(\partial\phi_0/\partial\mu)_{\zeta=\zeta_0}=-A/(\zeta_0{}^2-1)$ and hence the expression for $(\partial\phi/\partial\mu)_{\zeta=\zeta_0}$ becomes after simplying

$$\left(\frac{\partial\phi}{\partial\mu}\right)_{\zeta=\zeta_0}=\left(\frac{\partial\phi_0}{\partial\mu}\right)_{\zeta=\zeta_0}[1+M^2\sum_{n=0}^{n=\infty}P_{2n+1}{}'(\mu)H_{2n+1}(\zeta_0)] \quad . \quad . \quad (27)$$

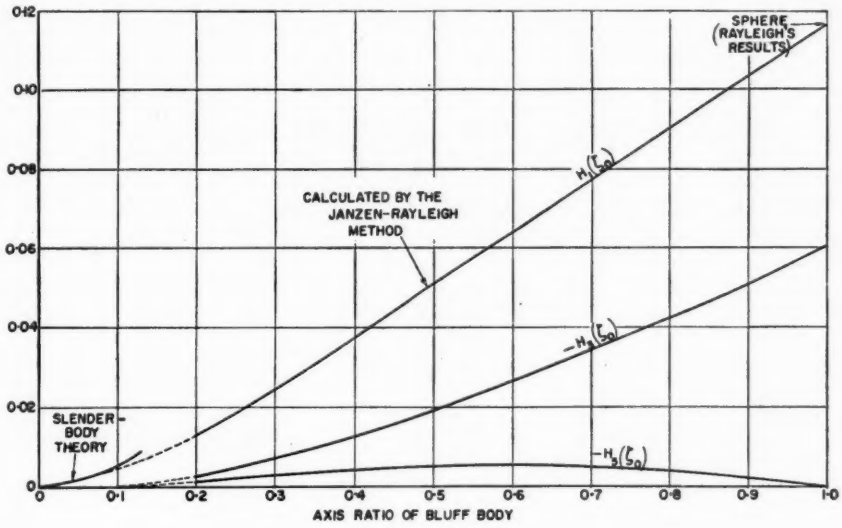

Fig. 1.

The functions $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ plotted against axis ratio.

$$\left.\begin{aligned} \text{where } H_1(\zeta_0) &= \frac{-3}{4Q_1'(\zeta_0)^3}\int_{\infty}^{\zeta_0} \frac{Q_1^2(\xi)(\xi^2+1)}{\xi^2(\xi^2-1)^2}\,d\xi - \frac{1}{2Q_1'^3}\left[\frac{5}{2}Q_1^2Q_1' - \frac{3Q_1^3}{2\zeta_0} + \right. \\ &\left. + Q_0^2Q_1' - \zeta_0Q_1Q_1'^2 - \frac{4}{3}Q_0^3 + \frac{5}{2}Q_1' + \frac{3Q_0}{\zeta_0^2-1} + Q_0Q_1'^2\right] \end{aligned}\right\} \quad . \quad (28)$$

and

$$H_{2n+1}(\zeta_0) = \frac{-(4n+3)}{4}\frac{1}{Q_1'^2Q_{2n+1}'}\left\{\int_{\infty}^{\zeta_0}\frac{Q_{2n+1}^2(\xi)(\xi^2+1)}{\xi^2(\xi^2-1)^2}\,d\xi - \frac{Q_{2n+1}^2(\zeta_0)}{\zeta_0(\zeta_0^2-1)}\right\}$$

for $n=1, 2\ldots$

The value of $q_{b,M}$, the fluid velocity at a point on the surface of the body, is

$$q_{b,M} = \frac{(1-\mu^2)^{\frac{1}{2}}}{K(\zeta_0^2-\mu^2)^{\frac{1}{2}}}\left(\frac{\partial\phi}{\partial\mu}\right)_{\zeta=\zeta_0}.$$

Hence if $q_{b,0}$ is the velocity of the fluid at the same point on the body when the fluid is supposed incompressible, then from equation (27) we have

$$q_{b,M} = q_{b,0}\left[1 + M^2\sum_{n=0}^{n=\infty}\{H_{2n+1}(\zeta_0)P_{2n+1}'(\mu)\}\right]. \quad . \quad . \quad . \quad (29)$$

Values of $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ for varying ζ_0 have been computed and these are given in Table I and also plotted in Fig. 1.

TABLE I

Axis ratio of the body	ζ_0	$H_1(\zeta_0)$	$H_3(\zeta_0)$	$H_5(\zeta_0)$
1	∞	0·117	−0·061	0
0·81	1·70	0·092	−0·044	−0·0039
0·60	1·40	0·065	−0·027	−0·0056
0·42	1·10	0·040	−0·014	−0·0048
0·20	1·05	0·013	−0·0027	−0·0016

4. Behaviour of $H_{2n+1}(\zeta_0)$ for Large Values of n

It is known[9] that $$Q_n(\mu) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \exp(-x)$$

for large values of n, where $x = n(\mu^2-1)^{\frac{1}{2}}$ and $\mu > 1$. Hence, applying this result we have for large n

$$Q_{2n+1}(\zeta_0) \sim \left[\frac{\pi}{2(2n+1)}\right]^{\frac{1}{2}} \frac{1}{(\zeta_0^2-1)^{\frac{1}{4}}} \exp[-(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}] \quad . \quad (30)$$

where ζ_0 is real, fixed and greater than unity.

Consider $\displaystyle\int_{\zeta_0}^{\infty} \frac{Q_{2n+1}^2(\xi)(\xi^2+1)\,d\xi}{\xi^2(\xi^2-1)^2}$ and denote this by $I_{2n+1}(\zeta_0)$.

Then $$I_{2n+1}(\zeta_0) \sim \int_{\zeta_0}^{\infty} \frac{\pi}{2(2n+1)} \frac{\xi^2+1}{\xi^2(\xi^2-1)^{5/2}} \exp[-2(2n+1)(\xi^2-1)^{\frac{1}{2}}]\,d\xi = I(\zeta_0)$$

$$< \frac{\pi}{2(2n+1)} \frac{(\zeta_0^2+1)}{\zeta_0^3(\zeta_0^2-1)^2} \int_{\zeta_0}^{\infty} \frac{\xi}{(\xi^2-1)^{\frac{1}{2}}} \exp[-2(2n+1)(\xi^2-1)^{\frac{1}{2}}]\,d\xi$$

$$= \frac{\pi}{4(2n+1)^2} \frac{(\zeta_0^2+1)}{(\zeta_0^2-1)^2\zeta_0^3} \exp[-2(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}].$$

Hence $$I_{2n+1}(\zeta_0) \sim I(\zeta_0) < \frac{\pi}{4(2n+1)^2} \frac{(\zeta_0^2+1)}{\zeta_0^3(\zeta_0^2-1)^2} \exp[-2(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}]. \quad (31)$$

Further, we have

$$\frac{Q_{2n+1}^2(\zeta_0)}{\zeta_0(\zeta_0^2-1)} \sim \frac{\pi}{2(2n+1)} \frac{1}{\zeta_0(\zeta_0^2-1)^{3/2}} \exp[-2(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}] \quad . \quad (32)$$

and hence from (31) and (32) it is seen that for large values of n the term in the expression for $H_{2n+1}(\zeta_0)$ which involves the integral may be neglected compared with the other term in the expression, these terms being at least in the ratio of $1/n : 1$.

Differentiating (30) with respect to ζ_0 we have, for large values of n,

$$Q_{2n+1}'(\zeta_0) \sim -\left[\frac{\pi(2n+1)}{2}\right]^{\frac{1}{2}} \frac{\zeta_0}{(\zeta_0^2-1)^{\frac{1}{4}}} \exp[-(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}]. \quad . \quad (33)$$

Hence, using the results of (31), (32) and (33), it follows that

$$H_{2n+1}(\zeta_0) \sim \frac{-(4n+3)}{4Q_1'^2(\zeta_0)} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{(2n+1)^{3/2}} \frac{1}{\zeta_0^2(\zeta_0^2-1)^{\frac{1}{4}}} \exp[-2(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}]. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The coefficients $H_{2n+1}(\zeta_0)$ are thus seen to behave like $n^{-\frac{1}{2}} \exp[-(2n+1)(\zeta_0^2-1)^{\frac{1}{2}}]$ for large values of n.

5. Thin Ellipsoids of Revolution

For the incompressible flow of a steady stream of fluid with velocity U past the ellipsoid of revolution $\zeta=\zeta_0$, the velocity potential ϕ_0 is given by[7]

$$\phi_0 = -UK\mu\zeta + \frac{UK\mu Q_1(\zeta)}{Q_1'(\zeta_0)}. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The perturbation potential ϕ_p on the steady stream flow due to the presence of the ellipsoid is therefore given by

$$\phi_p = \frac{UK\mu Q_1(\zeta)}{Q_1'(\zeta_0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If (u_1, v_1) are the components of the perturbation velocity in the Ox and radial directions respectively then

$$u_1 = \frac{-(1-\mu^2)^{\frac{1}{2}}}{K(\zeta_0^2-\mu^2)^{\frac{1}{2}}} \left(\frac{\partial\phi_p}{\partial\mu}\right)_{\zeta=\zeta_0} \cos\lambda,$$

where λ is the angle that the tangent to a meridian section of the body makes with the Ox axis. Clearly by using (6) it follows that $\cos\lambda = -\zeta_0(1-\mu^2)^{\frac{1}{2}}/(\zeta_0^2-\mu^2)^{\frac{1}{2}}$, and hence

$$\frac{u_1}{U} = \frac{Q_1(\zeta_0)}{Q_1'(\zeta_0)} \frac{\zeta_0(1-\mu^2)}{(\zeta_0^2-\mu^2)} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For a slender body ζ_0 is approximately unity (for example for a body of axis ratio 1 : 10, $\zeta_0 = 1{\cdot}005$). Hence write $\zeta_0 = 1+\delta$ where δ is small. Substituting in (37),

$$\frac{u_1}{U} = \frac{\left[\frac{1+\delta}{2}\log\frac{2+\delta}{\delta} - 1\right](1+\delta)(1-\mu^2)}{\left[\frac{1}{2}\log\frac{2+\delta}{\delta} - \frac{1+\delta}{2\delta+\delta^2}\right]\left[(1+\delta)^2-\mu^2\right]}$$

$$= \frac{\left\{2\delta - \delta\log\frac{2}{\delta} + O[\delta^2(\log\delta)^2]\right\}(1-\mu^2)}{(1-\mu^2+2\delta+\delta^2)}. \quad . \quad . \quad (38)$$

For slender bodies the axis ratio β/α is of the order of 0·1 or less and hence δ is less than or equal to 0·005. Thus

$$\frac{\delta}{1-\mu^2} \leqslant \frac{0{\cdot}005}{1-\mu^2} \leqslant 0{\cdot}1 \text{ if } 1-\mu^2 \geqslant 0{\cdot}05, \text{ that is if } \mu \leqslant 0{\cdot}97.$$

Hence certainly for $\mu \leqslant 0{\cdot}95$ we may expand in powers of $\delta/(1-\mu^2)$ and the result will be sufficiently accurate when we neglect squares and higher powers of this quantity. Thus, neglecting δ^2 compared with δ in the denominator of (38), and then expanding the denominator in powers of $2\delta/(1-\mu^2)$, we have

$$\frac{u_1}{U} = \left\{ 2\delta - \delta \log \frac{2}{\delta} + O\left[\delta^2 (\log \delta)^2\right] \right\} \left[1 - \frac{2\delta}{(1-\mu^2)} + O\left\{ \frac{\delta^2}{(1-\mu^2)^2} \right\} \right]. \quad (39)$$

Since $\zeta_0 = 1/e = 1+\delta$, then $\beta^2/\alpha^2 = 1-e^2 = 1-(1+\delta)^{-2} \simeq 2\delta$. Thus $\delta \simeq \beta^2/2\alpha^2$ and, on writing $t = \beta/\alpha$, equation (39) becomes

$$\frac{u_1}{U} = t^2 \left(1 - \log \frac{2}{t} + O\left[t^2 (\log t)^2\right] \right) \left(1 - \frac{t^2}{1-\mu^2} + O\left[\frac{t^4}{(1-\mu^2)^2} \right] \right). \quad (40)$$

If the equation of a section of the ellipsoid is $x^2/\alpha^2 + r_0^2/\beta^2 = 1$, then since $x = \alpha\mu$, we have $1-\mu^2 = r_0/\beta^2$. But $1-\mu^2 \geqslant 0{\cdot}05$ and therefore, $r_0^2/\beta^2 \geqslant 0{\cdot}05$. Hence $t^2/(1-\mu^2)$ is at most of order t. Equation (40) then becomes

$$\frac{u_1}{U} = t^2 \left(1 - \log \frac{2}{t} \right) + O(t^3 \log t). \quad . \quad . \quad . \quad . \quad (41)$$

The boundary condition on the ellipsoid is

$$\frac{v_1}{U-u_1} = -\tan \lambda = \frac{t\beta}{r_0} \left(1 - \frac{r_0^2}{\beta^2} \right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (42)$$

But $u_1 = O(Ut^2 \log t)$ and hence

$$v_1 = \frac{Ut\beta}{r_0} \left(1 - \frac{r_0^2}{\beta^2} \right)^{\frac{1}{2}} + O(Ut^{5/2} \log t). \quad . \quad . \quad . \quad (43)$$

since $(t\beta/r_0)(1-r_0^2/\beta^2)^{\frac{1}{2}}$ is at most of order $t^{\frac{1}{2}}$. If $q_{b,0}$ is the speed of the fluid at a point on the body surface, then from (41) and (43)

$$\begin{aligned} q_{b,0}^2 &= (U-u_1)^2 + v_1^2 \\ &= U^2 \left[\left\{ 1 - t^2 \left(1 - \log \frac{2}{t} \right) + O(t^3 \log t) \right\}^2 + \left\{ \frac{t\beta}{r_0} \left(1 - \frac{r_0^2}{\beta^2} \right)^{\frac{1}{2}} + O(t^{5/2} \log t) \right\}^2 \right] \\ &= U^2 \left[1 - 2t^2 \left(1 - \log \frac{2}{t} \right) + \frac{t^2\beta^2}{r_0^2} \left(1 - \frac{r_0^2}{\beta^2} \right) + O(t^3 \log t) \right]. \end{aligned}$$

Hence

$$1 - \frac{q_{b,0}^2}{U^2} = -2t^2 \log \frac{2}{t} + t^2 \left(3 - \frac{\beta^2}{r_0^2} \right) + O(t^3 \log t). \quad . \quad . \quad (44)$$

If $(C_p)_0$ is the pressure coefficient and is defined as

$$(C_p)_0 = \frac{(p-p_\infty)}{\frac{1}{2}\rho_\infty U^2},$$

where p_∞ is the pressure in the free stream, then

$$(C_p)_0 = 1 - \frac{q_{b,0}^2}{U^2} = -2t^2 \log \frac{2}{t} + t^2\left(3 - \frac{\beta^2}{r_0^2}\right) + O(t^3 \log t). \quad . \quad (45)$$

Using the result for slender bodies[10] that

$$(C_p)_{M,\beta/\alpha} = \frac{1}{(1-M^2)} (C_p)_{0,\beta(1-M^2)^{\frac{1}{2}}/\alpha} \quad . \quad . \quad . \quad . \quad (46)$$

which relates the compressible flow past a body of axis ratio β/α with the incompressible flow past a body of axis ratio $\beta(1-M^2)^{\frac{1}{2}}/\alpha$, $(C_p)_{M,\alpha/\beta}$ and $(C_p)_{0,\beta(1-M^2)^{\frac{1}{2}}/\alpha}$ being the pressure coefficients referred to these two flows, then we have from equation (44) the result

$$1 - \frac{q_{b,M}^2}{U^2} = -2t^2 \log \frac{2}{t(1-M^2)^{\frac{1}{2}}} + t^2\left(3 - \frac{\beta^2}{r_0^2}\right) + O(t^3 \log t), \quad . \quad (47)$$

where $q_{b,M}$ is the speed of the fluid at a point on the slender body of axis ratio β/α when the fluid is compressible. From (44) and (47) it follows at once that

$$\frac{q_{b,M}^2}{q_{b,0}^2} \simeq \frac{1 + 2t^2 \log\{2/[t(1-M^2)^{\frac{1}{2}}]\} - t^2(3-\beta^2/r_0^2)}{1 + 2t^2 \log(2/t) - t^2(3-\beta^2/r_0^2)}$$
$$= 1 - t^2 \log(1-M^2) + O(t^3).$$

Hence we have

$$q_{b,M}^2 \simeq q_{b,0}^2 [1 - t^2 \log(1-M^2)],$$

and expanding in powers of M^2 and neglecting M^4 and higher powers we have, to the required degree of approximation,

$$q_{b,M}^2 = q_{b,0}^2 [1 + M^2 t^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and this gives $$q_{b,M} = q_{b,0}\left[1 + \frac{1}{2}M^2 t^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Comparing this result with the general form of the result for bluff bodies, namely,

$$q_{b,M} = q_{b,0}[1 + M^2 F(\mu)] \quad . \quad . \quad . \quad . \quad . \quad (50)$$

it is seen that for slender ellipsoids $F(\mu)$ is a constant, in fact $F(\mu) = t^2/2 = \beta^2/(2\alpha^2) = \delta$. It must be remembered that this result has been shown only for the

range $\mu \leqslant 0{\cdot}95$, that is for $x \leqslant 0{\cdot}95a$. For points nearer the end of the body the slope of the body section increases rapidly and finally, at $x = a$, becomes infinite. This factor renders invalid the method here adopted, as it is necessary for the slope of the body surface to be small for the method to hold. The result obtained in equation (49) is easily obtained by the usual slender-body theory using source-sink distributions on the body axis.

6. Discussion of Results

It has already been stated in the Introduction that the form of the solution found here is

$$\phi = \phi_0 + M^2\phi_1 \quad \text{where} \quad \phi_1 = \sum_{n=0}^{n=\infty} b_{2n+1} P_{2n+1}(\mu),$$

$\mu = x/a$, the coefficients b_{2n+1} which are independent of μ and $P_{2n+1}(\mu)$ being the Legendre coefficients of the first kind. The fluid velocity $q_{b,M}$ at a point S on the surface of the body is calculated and it is given by

$$q_{b,M} = q_{b,0}\left[1 + M^2 \sum_{n=0}^{n=\infty} \left\{ H_{2n+1}(\zeta_0) \frac{d}{d\mu} P_{2n+1}(\mu) \right\}\right], \qquad (51)$$

where ζ_0 is the parameter denoting the particular spheroid considered (this being one of a family of confocal prolate spheroids) and $q_{b,0}$ is the velocity which the fluid at S would possess assuming it to be incompressible. Hence

$$\left[1 + M^2 \sum_{n=0}^{n=\infty} \left\{ H_{2n+1}(\zeta_0) \frac{d}{d\mu} P_{2n+1}(\mu) \right\}\right]$$

is a correction factor by which $q_{b,0}$ must be multiplied in order to take into account the effect of compressibility. The values of the coefficients $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ occurring in equation (50) have been calculated for some values of ζ_0 (that is for bodies of various axis ratios) and are plotted against axis ratio. These results are given later. The behaviour of $H_{2n+1}(\zeta_0)$ for large n is given, and this coefficient is found to decrease exponentially.

The parameter ζ_0 in the co-ordinates here used defines a prolate spheroid. For the whole family of confocal prolate spheroids ζ_0 varies in value between unity and infinity. When $\zeta_0 = 1$ the ellipsoid degenerates into a straight line joining the foci of the family, and when ζ_0 tends to infinity the ellipsoid tends to a sphere of infinite radius. However, the case of $\zeta_0 = 1$ is clearly of no interest and thus in all cases considered here ζ_0 will be greater than one. Furthermore, the case of a sphere has been discussed by Rayleigh[2] and hence the results when ζ_0 tends to infinity can be obtained without using the present method. For an ellipsoid of axis ratio 4 : 5, ζ_0 is only of the order of 1·7, so no large values of ζ_0 need enter into the computations. As ζ_0 approaches unity, computation becomes more laborious as the integrals required have a singularity at $\zeta_0 = 1$. Even for bodies of axis ratio 1 : 5, when $\zeta_0 = 1{\cdot}02$, the calculations are quite tedious and this factor limits the number of results which are reasonably obtainable for bodies of smaller axis ratio. Furthermore, as the axis ratio decreases it becomes necessary to calculate more coefficients

TABLE II

VALUES OF THE FUNCTIONS PLOTTED IN FIGS. 2 AND 3

$\mu=1$ or $s/a=0$ give the stagnation point on the nose of the body, the arc length s being measured from this point

Sphere	μ	1	0·9	0·8	0·6	0·4	0·2	0
Axis Ratio	s/a	0	0·4538	0·6458	0·9258	1·1606	1·3701	1·5708
=1·00	$F_0(\mu)$	−0·249	−0·1621	−0·0843	0·0438	0·1353	0·1902	0·2085
Ellipsoid	μ	1	0·9	0·8	0·6	0·4	0·2	0
Axis Ratio	s/a	0	0·3732	0·5393	0·7931	1·0182	1·2239	1·4245
=0·81	$F_0(\mu)$	−0·2305	−0·1344	−0·0579	0·0488	0·1103	0·1413	0·1507
Axis Ratio	μ	1	0·9	0·8	0·6	0·4	0·2	0
=0·60	s/a	0	0·2854	0·4287	0·6590	0·8755	1·0770	1·2774
	$F_0(\mu)$	−0·181	−0·0946	−0·0308	0·0464	0·0805	0·0924	0·0950
Axis Ratio	μ	1	0·9	0·8	0·6	0·4	0·2	0
=0·42	s/a	0	0·2149	0·3360	0·5495	0·7586	0·9591	1·1593
	$F_0(\mu)$	−0·116	−0·0560	−0·0120	0·0351	0·0505	0·0525	0·0520
Axis Ratio	μ	1	0·9	0·8	0·6	0·4	0·2	0
=0·20	s/a	0	0·1439	0·2463	0·4473	0·6504	0·8510	1·0510
	$F_0(\mu)$	−0·0271	−0·0095	0·0022	0·0138	0·0160	0·0149	0·0141
Axis Ratio	μ	1	—	—	—	—	—	0
=0·10	s/a	0	—	—	—	—	—	1·0232
	$F_0(\mu)$	is a constant and has the value 0·005						

in the series given in (50), for the further coefficients become increasingly comparable with the first three. Thus, for bodies of axis ratio less than about 2 : 5 more coefficients than three should be found and, in addition, the calculation of each is worsening, so that the results given will be useful chiefly for bodies of axis ratio greater than 0·4. Accordingly, the functions $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ as given by equation (28) have been calculated for values of $\zeta_0=1{\cdot}05$, 1·10, 1·40 and 1·70. These values correspond to bodies of axis ratio 0·20, 0·42, 0·60 and 0·81 respectively, all these figures being given correct to two decimal places. These results, together with the case of the sphere[2] are given in Table I and plotted in Fig. 1.

The case of the ellipsoid becoming slender is considered as the limiting form of the solution obtained in (50). ζ_0 is replaced by $1+\delta$ where δ is small and so $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ are obtained for bodies of axis ratio 0·1 and less. Combining these results with the ones already found and using Rayleigh's result for a sphere[2], complete curves of these three coefficients for bodies of axis ratio from 0 to 1 can be drawn. A reasonably smooth join is obtained between the portion of the graph representing the slender-body results and that giving the bluff-body results (Fig. 1).

If the general form of the velocity $q_{b,M}$ at a point S on the body is

$$q_{b,M}=q_{b,0}\,[1+M^2F(\mu)],$$

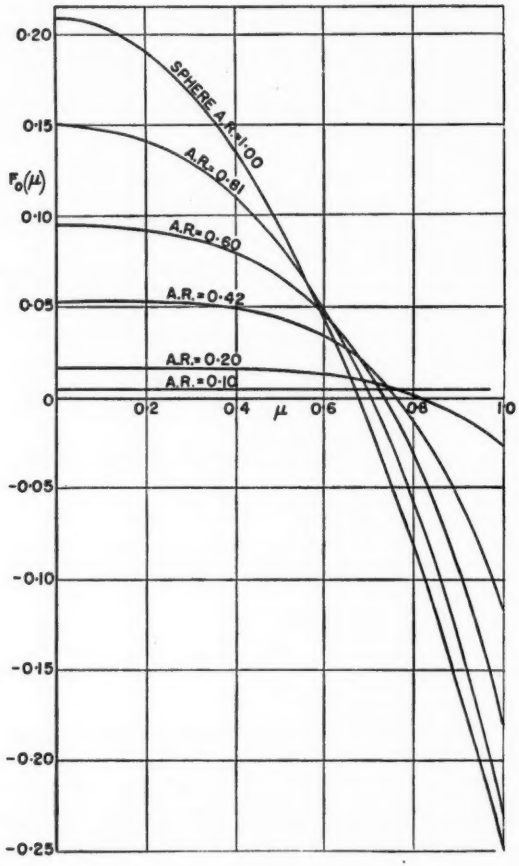

Fig. 2.
The function $F_0(\mu)$ plotted against μ for bodies of different axis ratio (A.R.). $\mu = 1$ gives the stagnation point at the nose of the body.

then Fig. 2 shows the function $F_0(\mu)$ plotted against μ for bodies of certain axis ratios, where $F_0(\mu)$ is an approximation to $F(\mu)$, namely, the first three terms in the series for $F(\mu)$. The results are also displayed in Fig. 3 by plotting $F_0(\mu)$ against s/a, where s is the arc length of a meridian section of the body measured from the forward stagnation point and a is the semi-major axis of the body; s is, of course, given by an elliptic integral. We have

$$ds = \frac{a}{\zeta_0}\left(\frac{\zeta_0^2 - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} d\mu$$

and hence

$$s = \frac{a}{\zeta_0}\int_1^{\mu}\left(\frac{\zeta_0^2 - \theta^2}{1-\theta^2}\right)^{\frac{1}{2}} d\theta. \quad . \quad . \quad . \quad . \quad (52)$$

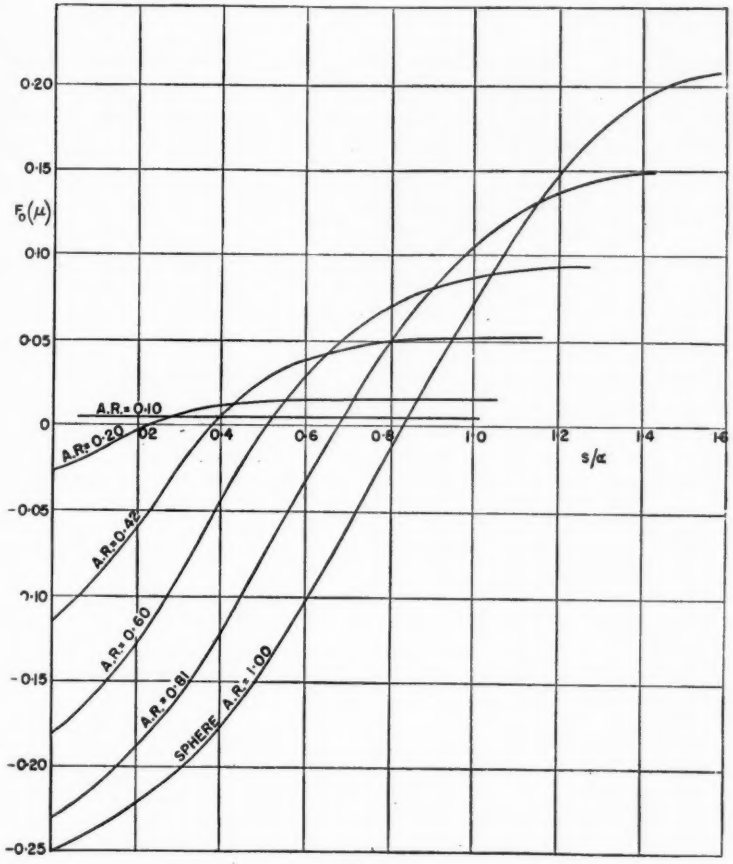

Fig. 3.
The function $F_0(\mu)$ plotted against s/α for bodies of different axis ratio (A.R.).

Using Fig. 1 it is possible by interpolation to read off the values of the coefficients $H_1(\zeta_0)$, $H_3(\zeta_0)$ and $H_5(\zeta_0)$ for a given body, that is, given the axis ratio of the ellipsoid which best approximates to the body, and then to plot $F_0(\mu)$ for that body. Alternatively, it is possible to interpolate for $F_0(\mu)$ from the graphs of this function which are given. As an example of the latter method, consider the case of a body of axis ratio 0·5. The value of $F_0(\mu)$ will be approximately $-0{\cdot}15$ when $s/\alpha=0$, it will be 0 when $s/\alpha=0{\cdot}45$ and will rise to $+0{\cdot}07$ when s/α attains its maximum value for this body.

ACKNOWLEDGMENT

The author wishes to express his thanks to Professor M. J. Lighthill for suggesting this problem and for the advice and encouragement he has given during the investigation of it.

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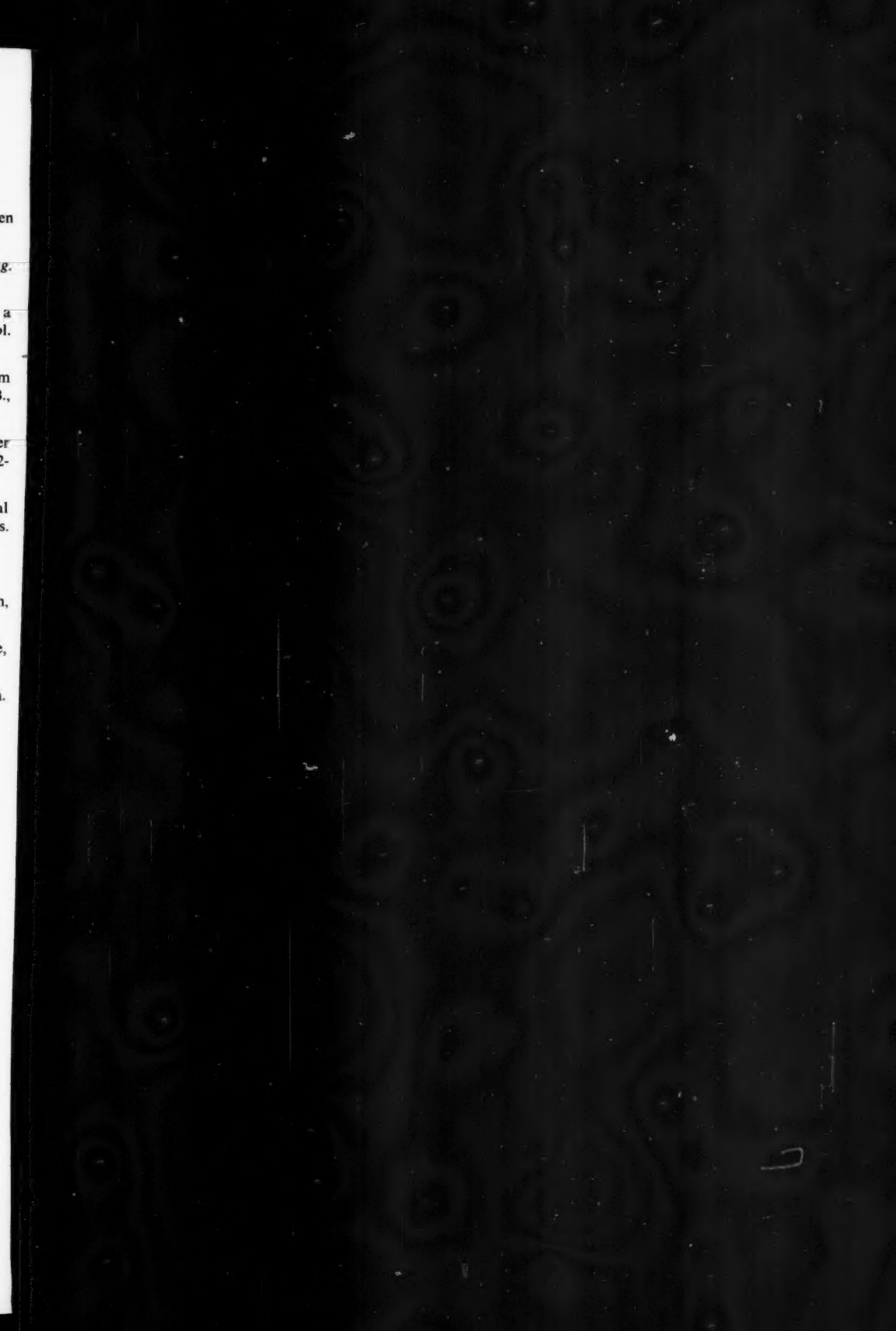
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